

Topology, Vol. 20, pp. 389–410, 1981
Printed in Great Britain. All rights reserved.

0040-9383/81/040389-22\$02.00/0
Copyright © 1981 Pergamon Press Ltd.

THE TOPOLOGICAL UNIQUENESS OF MINIMAL SURFACES IN THREE DIMENSIONAL EUCLIDEAN SPACE

WILLIAM H. MEEKS III

(Received 25 July 1979)

RECENTLY exciting progress has been made on the topological properties of minimal surfaces. The most important of the new results concern embedding properties of least area surfaces in three dimensional manifolds. For example, the embedding theorems of Meeks–Yau have been applied to classify compact group actions on three dimensional euclidean space, to prove the embedding of the classical Plateau's problem for an extremal Jordan curve in \mathbb{R}^3 , to prove the Bridge Principle, to prove extremal Jordan curves with total curvature less than or equal to 4π bound a unique compact branched minimal surface and to prove other interesting theorems in three dimensional topology and geometry (see [7–13, 21]). Also, quite recently, the author and de Melo Jorge in [3] have proved a large number of new theorems concerning the topology of complete minimal surface of finite total curvature.

This paper concerns the question of the uniqueness of the topological placement of certain minimal surfaces in euclidean three space. This research on minimal surfaces has led to the discovery of more general results in the classical theory of surfaces in euclidean space. The main result of the paper is Theorem 1 which is a statement about the topological uniqueness of certain compact surfaces with boundary. Theorem 1 is proved using elementary ideas from Morse theory and geometric topology. As easy corollary to Theorem 1 is the following

COROLLARY. Suppose M_1 and M_2 are compact homeomorphic embedded minimal surfaces in \mathbb{R}^3 . If the boundary of M_1 and of M_2 is a fixed extremal Jordan curve, then M_1 and M_2 are isotopic. Nontrivial examples of minimal surfaces exhibiting the properties in this corollary appear at the end of §2.

Numerous other applications of Theorem 1 are given in §1. Some of these applications also follow from the work of Morton[14]. Morton's results were found independently and depend on the deep result of Waldhausen which states that Heegard splittings of the three-dimensional sphere are standard. In §2 we generalize the earlier stated corollary. However, the results in §2 do depend on Waldhausen's theorem.

In §3 we give a Morse theoretic analysis of the complement of a codimension-one submanifold of \mathbb{R}^n which has non-positive sectional curvature. As a corollary of this analysis, we give a positive solution to Conjecture 28 in [7] which states that a complete proper minimal surface in a complete simply connected three-dimensional manifold of non-positive sectional curvature disconnects the space into two components with free fundamental groups.

In §4 we extend the earlier stated corollary concerning the topological uniqueness of compact minimal surfaces whose boundary is a single extremal Jordan curve. We prove that if M is a complete proper embedded minimal surface in \mathbb{R}^3 which is

diffeomorphic to a compact surface punctured in one point or to the sphere punctured in two points, then M is standardly embedded up to isotopy.

All of the theorems that appear here in §1 and 2 are also contained in Chapter 5 of the author's book "Lectures on Plateau's Problem" [7]. The author refers the interested reader to his book for further discussion of minimal surfaces and the deep unsolved problems on the topological properties of minimal surfaces.

§1. TOPOLOGICAL UNIQUENESS OF CERTAIN SURFACES

THEOREM 1. *Suppose $\gamma_1, \gamma_2, \dots, \gamma_n$ are a collection of disjoint Jordan curves in the xy plane P_0 in \mathbb{R}^3 and α is a Jordan curve on the parallel plane $P_1 = (0, 0, 1) + P_0$. Let $H(x, y, z) = z$ be the height function in \mathbb{R}^3 . Suppose M_1 and M_2 are two smooth embedded compact connected homeomorphic surfaces contained in the region N between the parallel planes P_0 and P_1 and suppose these surfaces have boundary curves $\gamma_1, \gamma_2, \dots, \gamma_n, \alpha$. If $H|_{M_1}$ and $H|_{M_2}$ are Morse functions without critical points on the boundary and without critical points of index 2, then M_1 and M_2 are isotopic $\text{rel}(\partial N)$ through smooth surfaces without local maximums with respect to H .*

Proof. The theorem will be proved by a series of reductions of the problem and simplifications of the surfaces M_1 and M_2 . It is suggested that the reader who is not familiar with elementary Morse theory consult the first chapter of the excellent text [15] for an introduction to many of the constructions given here. We will now commence the proof of the theorem by showing we may make certain geometric assumptions about our surfaces M_1 and M_2 . The reader should note that isotopy in this proof will always mean isotopic through smooth surfaces without local maximums with respect to the height function $H: \mathbb{R}^3 \rightarrow \mathbb{R}$.

ASSUMPTION 1. *The critical points of $H|(M_1 \cup M_2)$ have distinct heights.*

Since there are only a finite number of critical points of $H|_{M_1}$ and of $H|_{M_2}$, after a small "vertical" isotopy of M_1 and M_2 , that is the identity outside a small neighborhood of the critical points, we may create new surfaces M'_1 and M'_2 with the following properties

- (1) M'_i is ambiently isotopic M_i for $i = 1, 2$.
- (2) $H|_{M'_i}$ is a Morse function with the same number and type of critical points as $H|_{M_i}$ for $i = 1, 2$.
- (3) The critical points of $H|(M'_1 \cup M'_2)$ have distinct heights.

The existence of M'_1 and M'_2 with the above properties follows from the non-degeneracy of the critical points of $H|_{M_i}$ for $i = 1, 2$.

ASSUMPTION 2. *Suppose $p \in M_i$ ($i = 1, 2$) is a critical point of index 1. If $\epsilon > 0$ is sufficiently small and $B_\epsilon(p)$ is the ϵ -ball centered at p , then $B_\epsilon(p) \cap M_i = (B_\epsilon(p) \cap \{(x, y, -x^2 - y^2) + p\} | x, y \in \mathbb{R})$.*

. By compactness of M_i and the fact that M_i is smooth and embedded it can be seen that the intersection of a small ball $B_\delta(p)$ with M_i is a graph over the xy plane. By the argument given in the proof of Morse's Lemma, we can pick a coordinate system (u, v) in a small neighborhood $U \subset M_i$ of p so that $H|_U = H(p) - u^2 - v^2$.

Suppose that δ is small enough so that $B_\delta(p) \cap M$ is contained in U . With an ambient isotopy that fixes p , does not introduce a new critical point, and is the identity

outside a ball $B_{2\epsilon}(p)$, we can move the surface $B_\delta(p) \cap M$ to be the graph of the function $-x^2 + y^2 + H(p)$, for some ϵ with $0 < \epsilon < \delta$. The proof of the existence of such an ambient isotopy is elementary and we leave the proof to the reader.

ASSUMPTION 3. M_1 and M_2 are minimal counterexamples. That is to say $H|M_1$ has the minimum number of critical points such that there exists an M_2 satisfying the hypothesis of the theorem and M_1 and M_2 do not satisfy the conclusion. Similarly once M_1 is chosen, a minimal M_2 is chosen for M_1 . In our choice of M_1 and M_2 we do not restrict ourselves to the Jordan curves $\gamma_1, \gamma_2, \dots, \gamma_n$ but pick a minimal M_1 that arises from all finite collections of disjoint Jordan curves in the plane P_0 .

ASSUMPTION 4. The Jordan curve α is the circle $\alpha = \{x, y, 1 | x^2 + y^2 = 1\}$ and $(H|M_i)^{-1}([1 - \epsilon, 1]) = \{(x, y, z) | x^2 + y^2 = 1, z \in [1 - \epsilon, 1]\}$ for some $\epsilon > 0$.

By the Jordan curve theorem the Jordan curve α is the boundary curve of a closed disk D . After a horizontal translation of M_1 and M_2 , we may assume that the point $(0, 0, 1)$ lies in the interior of the disk $D \subset P_1$ bounded by the Jordan curve α . Since there are no critical points of M_1 and M_2 in some neighborhood of P_1 , the intersection of M_i with the plane $H^{-1}(r)$ for r close to 1 is a smooth Jordan curve α_i' for $i = 1, 2$ which depends differentiably on r .

Since the curves α_i' depend in a differentiable way on r , there is some small $\delta > 0$ so that there is a unique geodesic $\sigma(\alpha_1'(u)) : [0, 1] \rightarrow H^{-1}(r)$ which minimizes the distance from $p \in \alpha_1'$ to the curve α_2' for $r \in [1 - 2\delta, 1]$. Since the geodesics $\sigma(p)$ are minimal and unique, they are also disjoint and vary continuously with respect to the point $\alpha_1'(u)$ and $r \in [1 - 2\delta, 1]$. We may also assume that the geodesics $\sigma(\alpha_1'(u))$ are also disjoint for u in some larger interval $[-\epsilon', 1 + \epsilon']$ for some $\epsilon' > 0$.

Recall that N is the region in R^3 between the planes P_0 and P_1 . Let $R_t : N \rightarrow N$ be a level preserving isotopy with respect to H and defined for $t \in [0, 1]$ such that $R_t(\alpha_1'(u)) = \sigma(\alpha_1'(u))(t)$ and $R_t|_{\partial N} = id_{\partial N}$.

The construction of such an R_t is straightforward. Thus after a level preserving isotopy we may assume that M_1 and M_2 agree in a small neighborhood of the plane P_1 . After a similarly defined isotopy we may assume that the curves $\alpha_1' = \alpha_2' = \alpha - (0, 0, 1 - r)$ for $r \in [1 - \delta, 1]$. After another straightforwardly defined level preserving diffeomorphism $F : N \rightarrow N$, we have

$$F(\alpha_1') = F(\alpha_2') = \{(x, y, r) | x^2 + y^2 = 1\}$$

and hence $F(M_1)$ and $F(M_2)$ satisfy the conditions in the assumption.

Suppose now that there is a diffeomorphism $h : N \rightarrow N$ with $h|_{\partial N} = id_{\partial N}$ and $h(F(M_1)) = F(M_2)$. Then $M_2 = F^{-1} \circ h \circ F(M_1)$, where $(F^{-1} \circ h \circ F)|_{\partial N}$ is the identity map on ∂N . Also the number of critical points of $H|(F(M_i))$ is equal to the number of critical points of $H|M_i$ since F is a level preserving diffeomorphism. Thus $F(M_1)$ and $F(M_2)$ are two minimal counterexamples given in Assumption 3 if M_1 and M_2 are minimal counterexamples. Putting these facts together, we have shown that we may make Assumption 4.

ASSUMPTION 5. The Jordan curve γ_k in $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is a circle centered at the $(0, m_k, 0)$ in P_0 for some integer m_k and the radius of γ_k is less than or equal the radius of the circle γ_{k+1} . Furthermore, $(H|M_i)^{-1}([0, \epsilon])$ consists of cylinders of height ϵ over the circles $\gamma_1, \gamma_2, \dots, \gamma_n$ for some $\epsilon > 0$ and $i = 1, 2$. We may also assume that when γ_i is contained in the interior of the disk D_{i+k} in P_0 bounded by γ_{i+k} , then γ_i lies in the left half of D_{i+k} .

The argument given in the previous assumption generalizes directly to prove that we may make this assumption for the minimal counterexamples.

ASSUMPTION 6. $H|M_2$ has some critical point.

If $H|M_2$ has no critical point, then by Assumption 3 $H|M_1$ also has no critical points. This immediately implies that $n = 1$ and M_1 and M_2 are cylinders which intersect each plane $H^{-1}(r)$ in a Jordan curve for $r \in [0, 1]$. Rather straightforward arguments show that M_1 and M_2 are isotopic in the region N between P_0 and P_1 . Thus we may assume that for minimal counterexamples, $H|M_2$ has some critical point.

With these assumptions we can begin the main steps in the proof of the theorem. Our proof will be by contradiction. Suppose now that the theorem fails and let M_1 and M_2 be fixed minimal counterexamples as given in Assumption 3.

ASSERTION 1. For $i = 1, 2$, $(H|M_i)^{-1}([1 - r, 1])$ is connected for every r .

Proof. Suppose $(H|M_i)^{-1}([1 - r, 1])$ is not connected. Let X be a connected component of $(H|M_i)^{-1}([1 - r, 1])$ which does not contain the circle α . Since M_i is compact and $(H|M_i)^{-1}([1 - r, 1])$ is a closed subset of M_i , the subset X is compact. Hence $H|X$ has a maximum at some point $p \in X$. It is readily seen that p is local maximum for $H|X$ and p is a point in the interior X . This contradicts our assumption that $H|M_i$ has no critical points of index 2 and thus proves Assertion 1.

ASSERTION 2. If M_1 and M_2 are planar domains, then $H|M_i$ for $i = 1, 2$ does not have a critical point of index 0.

Proof. If we can show that when M_1 has a critical point of index 0 that M_1 is isotopic to an M'_1 satisfying the hypothesis of the theorem with respect to H and so that $H|M'_1$ has two less critical points, we will have reduced the number of critical points of M_1 in its ambient isotopy class. This will show M_1 does not satisfy the minimal condition. Hence $H|M_1$, and similarly $H|M_2$, could only have critical points of index 1. Assume now that M_1 has a critical point of index 0.

Let $p \in M_1$ be the critical point of index 0 for $H|M_1$ such that $\lambda = H(p)$ has the greatest value. Now consider the intersection of the planes $H^{-1}(r)$ with the surface M_1 for all values of $r > \lambda$. Except for r a critical value of $H|M_1$ we have $H^{-1}(r) \cap M_1$ is a collection of disjoint Jordan curves which for close values $r > \lambda$ vary in a continuous way on the planes $H^{-1}(r)$. It is clear from the Morse lemma that for values r slightly greater than λ there is a Jordan curve α_r in $H^{-1}(r)$ which bounds a disk $D_1(r)$ on M_1 which contains the point p . Since α_r is a Jordan curve in a plane, α_r also bounds a disk $D_2(r)$ in the plane $H^{-1}(r)$.

By Assertion 1 the set $H^{-1}([\lambda, 1])$ is connected and hence for some smallest value of T the circle α_T will touch another circle $\beta_T \subset H^{-1}(T) \cap M_1$ at some critical point q for $H|M_1$ or else there is a first time T when two points on α_r for $r < T$ come together to form a figure ∞ or a circle inside a circle \odot in the plane $H^{-1}(T)$.

Case 1. At the critical level T two points on α_r for $r < T$ converge to form an embedded ∞ or \odot in the plane $H^{-1}(T)$. Let δ_1 and δ_2 be the two circles in ∞ or \odot . Since the critical point q has index 1, Assumption 2 shows that for any small value $r > T$ these circles separate to form two circles $\delta_1(r)$ and $\delta_2(r)$. Since $H^{-1}([r, 1]) \cap M_1$

is connected by Assumption 1, there is a Jordan arc $\tau_1: [0, 1] \rightarrow (H|M_1)^{-1}([r, 1])$ such that τ_1 is transverse to $\delta_1(r)$ and $\delta_2(r)$ on M_1 and $\tau_1(0) \in \delta_1(r)$ and $\tau_1(1) \in \delta_2(r)$. Since we can join any point on $\delta_1(r)$ or $\delta_2(r)$ with the point q in the set $(H|M_1)^{-1}([0, r])$, there is another Jordan arc $\tau_2: [0, 1] \rightarrow (H|M_1)^{-1}([0, r])$ which is transverse to $\delta_1(r)$ and $\delta_2(r)$ and which has $\tau_2(0) = \tau_1(1)$ and $\tau_2(1) = \tau_1(0)$. Hence the composite Jordan curve $\tau_1\tau_2: S^1 \rightarrow M_1$ intersects the circle $\delta_1(r)$ transversely in a single point. Since this implies M_1 is not a planar domain, case 1 cannot occur.

Case 2. The Jordan curves α_r for $r < T$ converge at one point p with some other family of Jordan curves $\beta_r \subset H^{-1}(r) \cap M_1$ to form topologically a ∞ or \odot in the plane $H^{-1}(T)$. The limiting Jordan curve α_T bounds a disk D_1 on the surface M_1 which contains the critical point p . Also α_T bounds a disk D_2 on the plane $H^{-1}(T)$. Now consider the part of M_1 contained in the interior of the ball B bounded by the sphere $D_1 \cup D_2$. If $(\text{int}(B)) \cap M_1$ is nonempty, consider a component A in $(\text{int}(B)) \cap M_1$. The height function $H|A$ certainly has a minimum at some point $a \in A$. Since $H(a) > H(p)$, this gives a contradiction to our choice of p as a critical point of index 0 with $H(p)$ having the greatest value. Hence the interior of B is disjoint from M_1 .

Since the interior of B is disjoint from M_1 , the disk D_1 is isotopic to D_2 in $(B - M)$ with boundary fixed. After smoothing off this isotopy along α_T and pushing the disk D_2 up a little bit, one can cancel the critical points p and q . In other words, after an ambient isotopy which takes place in an arbitrarily small neighborhood of the ball B , we have a new surface M'_1 , ambiently isotopic to M_1 , which satisfies the hypothesis of the theorem and such that the number of critical points of $H|M'_1$ is less than the number of critical points of $H|M_1$. This gives a contradiction to our minimal choice of M_1 . This contradiction completes the proof of Assertion 2.

ASSERTION 3. *The surface M_1 and M_2 are planar domains.*

Proof. Note that if M_1 is not a planar domain, then the first Betti number of M_1 is greater than n where n is the number of circles $\gamma_1, \dots, \gamma_n$ in the plane P_0 . Using the Morse inequalities, this fact about the first Betti number of M_1 implies that the number of critical points of $H|M_1$ is greater than $(n - 1)$. In fact the Morse inequalities imply that there is a critical point $p \in M_1$ for $H|M_1$ of index 1 so that the number of Jordan curves in $H^{-1}(\lambda + t) \cap M_1$ is greater than the number of Jordan curves $H^{-1}(\lambda - t) \cap M_1$ where $\lambda = H(p)$ and $0 < t < \epsilon$. Here ϵ is small enough so that the point p is the unique critical point for $H|(H|M_1)^{-1}([\lambda - \epsilon, \lambda + \epsilon])$.

Thus at the critical level λ , a component of the intersection of M_1 with the plane $H^{-1}(\lambda)$ is a figure ∞ or \odot where p is the singular point and this figure becomes two distinct circles α_r and β_r for $r \in (\lambda, \lambda + \epsilon]$. We will show that the figure ∞ case is impossible. A similar argument will prove that the other case is also impossible. The proof of this second possibility will be left to the reader.

The Jordan curves α_r and β_r bound disks $D_1(r)$ and $D_2(r)$ respectively in the planes $H^{-1}(r)$. Since $(H|M_1)^{-1}([r, 1])$ is connected, our minimal choice of M_1 implies that any surface M'_1 bounding the Jordan curves $H^{-1}(r) \cap M$ and α and which satisfies the conditions in the theorem is ambiently isotopic to $(H|M_1)^{-1}([r, 1])$ relative to the planes $H^{-1}(r)$ and $P_1 = H^{-1}(1)$. We will now choose a particular representative M'_1 .

First consider any "standard" examples of planar domains A_1 bounded by the curve $\alpha_{\lambda+\epsilon}$ and the curves in $D_1(r) \cap M_1$ and A_2 bounded by $\beta_{\lambda+\epsilon}$ and the Jordan curves in $D_2(r) \cap M_1$. Here $r \in (\lambda, \lambda + \epsilon)$ is fixed and A_1 and A_2 are assumed to satisfy the conditions in the theorem and to have the minimal possible number of critical

points. The planar domains A_1 and A_2 can be extended above the level $H^{-1}(\lambda + \epsilon)$ to be part of a surface M'_1 diffeomorphic to M_1 and having the minimal possible number of critical points and satisfying the hypothesis of the theorem with respect to the Jordan curves $H^{-1}(r) \cap M_1$ and α . By minimality of M_1 , M'_1 is ambiently isotopic to $(H|M_1)^{-1}([r, 1])$ relative to the planes $H^{-1}(r)$ and P_1 . This shows M_1 is isotopic to $M''_1 = M'_1 \cup (H|M_1)^{-1}([0, r])$ and $H|M''_1$ has no more critical points than $H|M_1$.

By making appropriate choices for the planar domains A_1 and A_2 given above, we can change the surface M''_1 by sliding the connections of the Jordan curves inside of $D_1(r)$ and $D_2(r)$ below the critical level λ by an ambient isotopy. In other words we may assume that for our original example M_1 there is a continuous family of Jordan curves $\delta_t \subset H^{-1}(t) \cap M_1$ for small $t < \lambda$ that converge to the Jordan curves $\alpha_\lambda \cup \beta_\lambda$ and such that the disk $D(t)$ bounded by δ_t on the plane $H^{-1}(t)$ has the property $D(t) \cap M_1 = \delta_t$ for $t \in (\lambda - \epsilon, \lambda)$.

Now consider the surface $M'_1 = (H|M_1)^{-1}([\lambda + \epsilon, 1])$ for ϵ sufficiently small so that $B_\epsilon(p) \cap M_1$ also satisfies Assumption 2. By minimality of M_1 , M'_1 is standardly embedded up to ambient isotopy. Consider the standard example which first connects the parts of the circles α_r and β_r lying in $B_\epsilon(p)$ for r close to λ . In fact this can be achieved by quickly shrinking the curves α_r and attaching the tube T formed by these new Jordan curves to the cylinder formed by the Jordan curves β_r (see Fig. 1 below).

Thus the surface M_1 is ambiently isotopic to a surface with a handle T attached to the cylinder formed by the δ_t curves and the β_r curves. One can shrink this handle by a diffeomorphism that preserves the level sets of H and is the identity outside the union of small neighborhoods of the disks $D_1(r)$ and $B_\epsilon(p)$. After shrinking T sufficiently we can slide this handle up along the cylinder formed by the Jordan curves β_r . In fact by choosing a Jordan arc $\tau: [0, 1] \rightarrow M_1$ with $\tau(0) \in \beta_{\lambda+\epsilon}$ and $\tau(1)$ very close to the curve α in P_1 , we can slide the small handle T within the neighborhood of P_1 where the surface M is given by vertical translations of the circle α (see Assumption 4).

Thus, after an ambient isotopy, the surface M_1 will move to a surface M'_1 with a trivial handle attached on the "outside" of the cylinder C formed by the vertical translates of α . We may assume this if the original singular component in $H^{-1}(\lambda) \cap M_1$ was ∞ . (If instead the singular component was \odot , then a similar proof gives an M'_1 with a trivial handle attached on the "inside" of C .) Whether one attaches a handle on the "inside" or the "outside" of C , the two possible surfaces are easily shown to be isotopic.

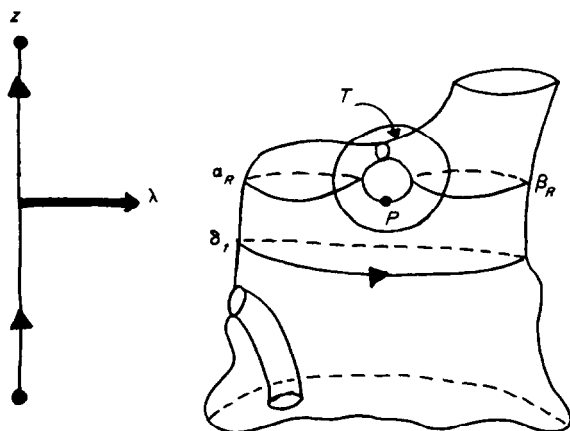


Fig. 1.

By the process of creating handles, one can assume after an isotopy that the surface M_1 consists of a cylinder C ϵ -close to P_1 with some number of handles trivially attached and that below this region one has the surface $(H|M_1)^{-1}([0, 1 - \epsilon])$ satisfying the conditions of the theorem relative to the planes P_0 and $H^{-1}(1 - \epsilon)$. Furthermore, the surface $M' = (H|M_1)^{-1}([0, 1 - \epsilon])$ is a planar domain with $H|M'$ having fewer critical points. This shows M' is standard and hence M_1 is isotopic to a "standard" example. A similar argument shows M_2 is isotopic to a "standard" example. Thus M_1 and M_2 are ambiently isotopic which proves Assertion 3.

From this point on, we can assume that M_1 and M_2 are planar domains. To prove that M_1 is isotopic to M_2 it is sufficient to show that M_1 and M_2 are ambiently isotopic to a "standard example" of a planar domain satisfying the hypotheses of the theorem. We will now give a definition of a standard example of a planar domain that is usable and for which it is easily shown that any two standard examples are ambiently isotopic.

First consider the cylinder C_1 of height $1/2$ over the circle γ_1 centered at $(0, 0)$ (see Assumption 5) which is given by the circles $\gamma_1 + (0, 0, t)$ for $t \in [0, 1/2]$. Next extend the cylinder in a differentiable way to join with the circle α (see Fig. 2). Next consider the cylinder C_2 of height $1/3$ over the circle γ_2 given by the circles $\gamma_2 + (0, 0, t)$ for $t \in [0, 1/3]$. Then join the cylinder C_2 directly to the cylinder C_1 by a thin tube T_1 so that it tilts somewhat in the direction $(0, -1, 0)$. Now continue this process of creating a cylinder C_k of height $1/(k+1)$ over the circle γ_k and then join C_k by a thin tube to the previous cylinder in the $(0, -1, 0)$ direction. At the stage n of this construction one has a "standard example" of a planar domain with boundary curves α and $\gamma_1, \gamma_2, \dots, \gamma_n$ which satisfies the hypotheses of the theorem.

We will now complete the proof of the theorem by showing that M_1 , and similarly M_2 is isotopic to a standard example. To see this consider the critical point $p \in M_1$ for $H|M_1$ with the smallest value $\lambda = H(p)$. Therefore $H^{-1}(r) \cap M_1$ is a collection of n disjoint Jordan curves $\gamma_1', \dots, \gamma_n'$ which vary differentiably for $r \in [0, \lambda)$. Here we will assume $\gamma_1^0 = \gamma_1, \dots, \gamma_n^0 = \gamma_n$ are the original Jordan curves in the plane P_0 .

Since the curves $\gamma_1', \dots, \gamma_n'$ vary smoothly with r , these curves arise from a one-parameter family of diffeomorphisms $f_r: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. In other words, $\gamma_k'(t) = (\pi(\gamma_1'(t)), r) = (f_r(\gamma_0(t)), r)$ where $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is orthogonal projection on the xy plane

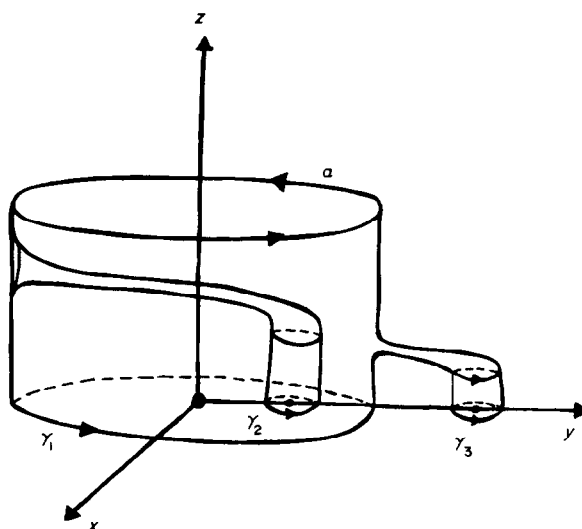


Fig. 2.

\mathbb{R}^2 . Because M_1 is a planar domain with all critical points of index one, two circles γ_i' and γ_j' come together as r approaches λ to form ∞ or \odot in the plane $H^{-1}(\lambda)$.

By Assumption 2 we may assume that ϵ is small enough so that $B_{2\epsilon}(p) \cap M_1$ is the graph given in Assumption 2. This assumption implies for r sufficiently close to λ , the Jordan curves γ_i' and γ_j' approach each other in a well-defined manner near the critical point p . Now consider the disk D' in \mathbb{R}^3 which is the intersection of the plane $P = \{(x, y_0, z) | x, z \in \mathbb{R}, y_0 \text{ is the } y \text{ coordinate of the point } p\}$ with the regions

$$R_1 = \{(x, y, z) | z \geq \lambda - \epsilon\}$$

and

$$R_2 = \{(x, y, z) + p | z \leq -x^2 + y^2\}$$

(see Fig. 3).

Let τ be the arc of intersection of D' with the plane $H^{-1}(\lambda - \epsilon)$. Note that the end points of τ are in the Jordan curves $\gamma_i^{\lambda - \epsilon}$ and $\gamma_j^{\lambda - \epsilon}$ and that the interior of τ lies in the complement of the curves $\gamma_1^{\lambda - \epsilon}, \dots, \gamma_n^{\lambda - \epsilon}$ on the plane $H^{-1}(\lambda - \epsilon)$. Now consider the continuous family of arcs given by $\tau_r = (f_r \circ f_{\lambda - \epsilon}^{-1}(\tau), r)$ for $r \in [0, \lambda - \epsilon]$. The union of these arcs τ_r together with the disk D' gives rise to a piecewise differentiable disk D'' . After a small perturbation of D'' near the arcs τ and τ_0 we may assume that there is a smooth disk D in \mathbb{R}^3 satisfying the following property.

(1) $D \cap H^{-1}(r)$ for $r \in [0, \lambda)$ is a Jordan arc τ_r that joins a point on γ_i' to a point on γ_j' and the interior of τ_r is disjoint from the other Jordan curves in $\gamma_1', \dots, \gamma_n'$ and $D = \bigcup_{r \in [0, \lambda)} \tau_r \cup \{p\}$.

After an appropriate isotopy of D through disks D_t satisfying property (1), we may choose D so that the isotopy class of τ_0 is simplest. To be precise, we may assume τ_0 satisfies the following conditions.

(2) The tangent vector $\tau_0'(t)$ is never zero and is parallel to the x axis the least possible number of times.

(3) τ_0 has the least number of intersections in its isotopy class as a map of pairs

$$\tau: ([0, 1], \{0, 1\}) \rightarrow \sqrt{(R^2 - \bigcup_{k=1}^n \gamma_k, \gamma_i \cup \gamma_j)}$$

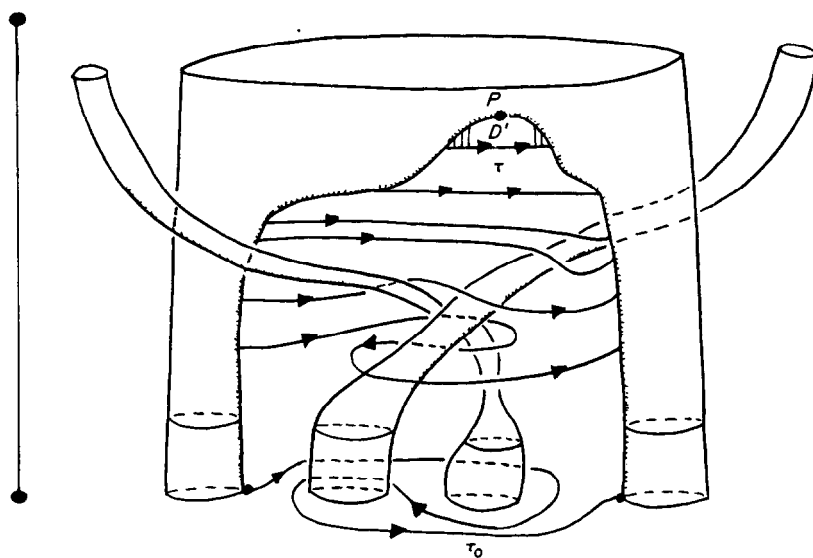


Fig. 3.

with the lines

$$L_k = \{(t, m_k, 0) | t \leq 0, m_k \text{ as in Assumption 5}\}$$

for $k = 1, 2, \dots, n$.

(4) The arc $\tau_r = \tau_0 + (0, 0, r)$ for $r \in [0, \epsilon]$ where ϵ is the ϵ given in Assumption 5.

We now give a proof of the following

ASSERTION 4. *After an isotopy of the minimal example M_1 , the disk D described above and the surface M_1 satisfy the following additional assumptions.*

(5) If $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ denotes orthogonal projection on the plane P_0 , then $\pi(D) = \tau_0$ and $(\pi|_{D^{-1}(x)})$ is a connected interval.

(6) There is a small $\delta > 0$ and $\delta < \epsilon$ so that the curves $\gamma_k' = \gamma_k + (0, 0, r)$ for $r \in [0, \lambda + 4\delta]$ for $k \neq i, j$.

(7) $\gamma_i' = \gamma_i + (0, 0, r)$ and $\gamma_j' = \gamma_j + (0, 0, r)$ for $r \in [0, \lambda - \delta]$ and $(\gamma_i \# \gamma_j)' = (\gamma_i \# \gamma_j) + (0, 0, r)$ for $r \in [\lambda + \delta, \lambda + 4\delta]$. Here $\gamma_i \# \gamma_j$ denote the connected sum of γ_i and γ_j along the arc τ_0 in the plane P_0 . The connected part of M_1 bounded by the curves $\gamma_i^{\lambda-\delta}$, $\gamma_j^{\lambda-\delta}$ and $(\gamma_i \# \gamma_j)^{\lambda+\delta}$ will hence forth be referred to as "the bridge" because of its bridgelike appearance. ((Note λ is the λ after the isotopy)).

Proof. Let X be the union of the curves γ_k' where $r \in [0, \lambda]$ and k is different from i and j . Since X and D are compact and disjoint, the distance from X to D is less than a small positive number $3d$.

Now consider the strip S which is a small regular neighborhood of the intersection of the disk D with the surface $H^{-1}[\epsilon/4, 1]$. After an isotopy of M which is the identity outside a $2d$ neighborhood of D and which slides the strip S down into the region $H^{-1}[\epsilon/4, \epsilon]$, we have a new disk which satisfies (5–7). In fact, the new disk is part of the previous disk D . See Fig. 4 for a surface M_1 and a disk D which satisfies this assertion.

ASSERTION 5. *Suppose D_k is the disk bounded by γ_k in the plane P_0 . If D_k is disjoint from γ_i and γ_j , then the interior of D_k is disjoint from the minimal example M_1 .*

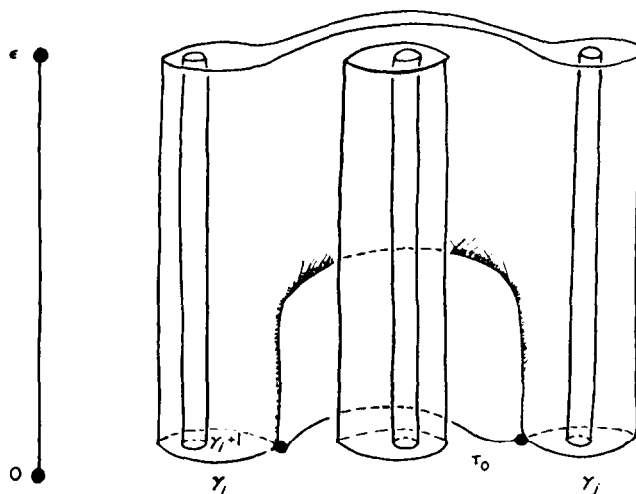


Fig. 4.

Proof. After an isotopy, assume that M_1 satisfies the conditions of assertion 4. Suppose that the curve γ_{k+1} is contained in the disk D_k and let $\gamma_{k+1}, \gamma_{k+2}, \dots, \gamma_{k+t}$ be the circles contained in the disk D_{k+1} bounded by γ_{k+1} . Let C be the cylinder over γ_k in the region $H^{-1}([0, \lambda + 4\delta])$.

Now consider the standard example A of a planar domain with boundary circles $\lambda_{k+1}^{\lambda+2\delta}$ and $\gamma_{k+1}^{\lambda+\delta}, \gamma_{k+2}^{\lambda+\delta}, \dots, \gamma_{k+t}^{\lambda+\delta}$ contained in the region between the planes $H^{-1}(\delta + \lambda)$ and $H^{-1}(\lambda + 2\delta)$. By minimality of M_1 , the surface $M'_1 = (H|M_1)^{-1}([\lambda + \delta, 1])$ is standard up to isotopy. Now construct a particular example of M'_1 with curves being α and $(H|M'_1)^{-1}(\lambda + \delta)$, which is made by first forming the planar domain A and then connecting the top circle $\gamma_{k+1}^{\lambda+2\delta}$ of A to the side of C by a thin tube T in the region $H^{-1}([\lambda + 2\delta, \lambda + 3\delta])$. Form T so that it projects orthogonally onto a small neighborhood of the set $\{[0, t, 0] | t \leq m_{k+1}\} \cup D_{k+1}$ in the plane P_0 . The connections of the other curves in $(H|M'_1)^{-1}(\delta + \lambda)$ are made by taking cylinders of height 3δ over these curves. Next join these cylinders by thin tubes of the cylinder C and then connect C to the curve α .

The surface $\tilde{M}_1 = M'_1 \cup (H|M_1)^{-1}([0, \delta + \lambda])$ is isotopic to M_1 and has a simple appearance in the region $H^{-1}(0, \lambda + 3\delta)$. Now slide the tube T together with the planar domain A down the side of the cylinder C and below the plane $H^{-1}(\lambda - \delta)$. After the end of this isotopy we have a new surface \bar{M}_1 which is isotopic to M_1 and in the region $H^{-1}([0, \lambda - \delta])$ has a very simple appearance. By minimality of \bar{M}_1 the surface $(H|\bar{M}_1)^{-1}([\lambda - \delta, 1])$ is isotopic relative to the planes $H^{-1}(\lambda - \delta)$ and P_1 to a standard embedded surface \bar{M}'_1 . It is straightforward to show that $\hat{M}_1 = \bar{M}'_1 \cup (H|\bar{M}_1)^{-1}([0, \lambda - \delta])$ is isotopic to a standard example. On the other hand, \hat{M}_1 is isotopic to M_1 and so M_1 is isotopic to a standard example contrary to our assumption about M_1 . This contradiction proves the assertion.

Using the same argument as in Assertion 5 one can prove the following and its proof will be left to the reader.

ASSERTION 6. *If D_j is contained in D_i , then the interior of D_j is disjoint from M_1 .*

Since the surface $(H|M_1)^{-1}([\lambda + 2\delta, 1])$ is standard up to isotopy, it is not difficult to prove that M_1 is isotopic to a standard example when the arc τ_0 is disjoint from the lines $L_k = \{(t, m_k, 0) | t \leq 0\}$ for $k = 1, 2, \dots, n$. We will leave this construction to the reader. What we will now prove is that one can decrease the number of points in

$$\tau_0 \cap \bigcup_{i=1}^n L_i$$

by an isotopy of the original M_1 . In other words, we will show that after an appropriate isotopy there is a new arc τ'_0 joining γ_i and γ_j where τ'_0 satisfies conditions (1-4) given earlier and

$$|\tau'_0 \cap \bigcup_{i=1}^n L_i| < |\tau_0 \cap \bigcup_{i=1}^n L_i|.$$

If we can prove the existence of τ' , then by induction we will have a proof of the theorem.

If some line L_k intersects τ_0 nontrivially, then we may assume there is another k , possibly equal to k' so that τ_0 intersects L_k and the disk D_k bounded by γ_k does not intersect the disk D_i or the disk D_j . Let x be the first point where L_k intersects τ_0 . Let

σ be the arc on L_k that joins the point $x \in \tau_0$ with the circle γ_k . Note that the surface $M'_1 = (H|M_1)^{-1}([\lambda + 2\delta, 1])$ is the unique planar domain (up to isotopy satisfying the conditions in the theorem) with these boundary curves. Also by Assertions 4–6 the surface $(H|M_1)^{-1}([0, \lambda + 2\delta])$ has a special appearance.

We will now consider a special example M''_1 of a planar domain with boundary curves $\Gamma = (H|M_1)^{-1}(\lambda + 2\delta)$ and α . In a neighborhood of the plane $H^{-1}(\lambda + 2\delta)$ we will assume that M''_1 is given by vertical translates of its boundary curves Γ . Now connect the various cylinders above the circles in Γ by thin tubes in the region $H^{-1}([\lambda + 3\delta, \lambda + 4\delta])$ to the cylinder C above $(\gamma_i \neq \gamma_j)^{\lambda + 2\delta}$ and then later connect this cylinder with α to produce a surface satisfying the appropriate conditions in the theorem. However, now vary this connecting procedure for the cylinder above $\gamma_k^{\lambda + 2\delta}$ as follows. In the region $H^{-1}([\lambda + 2\delta, \lambda + 3\delta])$, first connect $\gamma_k^{\lambda + 2\delta}$ to the cylinder C by a thin tube T which projects on to a small neighborhood of the arc σ in the plane P_0 together with the disk bounded by γ_k in this plane. Assume now that M_1 has the form described here.

Now slide the tube T together with the curves γ_k^r for $r \in [\lambda - \delta, \lambda + 2\delta]$ down the cylinder formed by the curves $(\gamma_i \neq \gamma_j)^r$ for $r \in [\lambda, \lambda + 3\delta]$ and then under the bridge formed by the circles $\gamma_i^r \cup \gamma_j^r$ for $r \in [\lambda - \delta, \lambda]$. Now slide this tube T up the “other” side of the cylinder C .

After the isotopy described in the previous paragraph, one has by direct construction that there is a new disk D' satisfying the conditions (1–4) and $D' \cap P_0$ is a curve τ'_0 satisfying the following additional properties

$$(a) \quad \tau'_0 \cap L_m = \tau_0 \cap L_m \text{ for } m \neq k$$

$$(b) \quad \tau'_0 \cap L_k = (\tau_0 \cap L_k) - \{x\}.$$

Thus after various isotopies we may assume that the arc τ_0 is disjoint from the lines L_1, \dots, L_n . As remarked earlier, this is sufficient to show M_1 is standardly embedded after an isotopy. This now implies Theorem 1.

COROLLARY 1. *Suppose α is a smooth Jordan curve on a plane P_0 in \mathbb{R}^3 and $\gamma_1, \gamma_2, \dots, \gamma_n$ are a disjoint collection of smooth Jordan curves on a parallel plane P_1 in \mathbb{R}^3 . Then any two embedded homeomorphic compact connected minimal surfaces M_1 and M_2 in \mathbb{R}^3 with boundary curves $\alpha, \gamma_1, \dots, \gamma_n$ differ by an ambient isotopy in the region between P_0 and P_1 that is the identity on $P_0 \cup P_1$.*

Proof. Clearly we may assume that P_0 is the xy plane and P_1 is a parallel plane. Consider the function $H(x, y, z) = z$ given in Theorem 1. Then if $H|M_1$ and $H|M_2$ are Morse functions, then because M_1 and M_2 have nonpositive curvature the functions $H|M_1$ and $H|M_2$ only have critical points of index 1. Thus in the case $H|M_1$ and $H|M_2$ have no degenerate critical points, Theorem 1 implies corollary 2.

If $H|M_1$ or $H|M_2$ has a degenerate critical point, then it is easily seen that this critical point is a zero of the Gauss curvature on M_1 or M_2 with normal vector $\pm(0, 0, 1)$. Since there are only a finite number of zeros of Gauss curvature on $M_1 \cup M_2$, one can perturb H in the region between P_0 and P_1 so that the level sets of the perturbed function H' are topologically planes and $H'|M_1$ and $H'|M_2$ are Morse functions. It is easily seen, using the proof of Theorem 1, that this situation implies M_1 and M_2 are isotopic and proves the corollary.

COROLLARY 2. *Suppose $g: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a smooth function such that the level sets $g^{-1}(r)$ are connected and topologically spheres or planes except possibly one level set is a point. If M is a compact connected embedded smooth surface in \mathbb{R}^3 without boundary and such that $H|M$ is a Morse function with exactly one critical point of index 2, then M is isotopic to the standard embedding.*

Proof. Suppose for the moment that the level sets of g consist of planes. By general position arguments we may assume that $g|M$ has exactly one global maximum $p \in M$, one global minimum $q \in M$. Consider the value $\lambda_1 = g(q)$ and the value $\lambda_2 = g(p)$. For $\epsilon > 0$ and sufficiently small $(g|M)^{-1}((\lambda_2 - \epsilon, \infty))$ is a disk D_1 and $(g|M)^{-1}((-\infty, \lambda_1 + \epsilon))$ is a disk D_2 . That these subsets of M are disks for some $\epsilon > 0$ follows immediately from the fact that p and q are nondegenerate critical points of index 2 and 0, respectively.

Now consider the surface $M' = M - (D_1 \cup D_2) = (g|M)^{-1}(\lambda_1 + \epsilon, \lambda_2 - \epsilon)$ lies in the region $R = g^{-1}(\lambda_1 + \epsilon, \lambda_2 - \epsilon)$. Since $g|M'$ is a Morse function on M' with no critical points of index 2, the proof of Theorem 1 shows that M' is standardly embedded in R up to isotopy. This immediately implies that the surface M is standardly embedded up to isotopy.

In the case that the level sets of g consist of a point and spheres, a similar argument yields a surface $M' = (g|M)^{-1}(\lambda_1 + \epsilon, \lambda_2 - \epsilon) = R$ and $g|M'$ has no critical points of index 2. In this case R is diffeomorphic with $S^2 \times [0, 1]$ and $g|R$ has no critical points. It is straightforward to construct a smooth Jordan arc $\alpha: [0, 1] \rightarrow R - M'$ so that α intersects each level set of $g|R$ in exactly one point. Thus the level sets of $g|(R - \alpha)$ now consist entirely of planes and the argument given in Theorem 1 shows that M' is standardly embedded up to isotopy in $R - \alpha$. This implies that M is standardly embedded in \mathbb{R}^3 . This proves Corollary 2.

COROLLARY 3. *Suppose M is a compact connected embedded surface in \mathbb{R}^3 of genus g . If the total curvature $\int_M |K| dA < 4\pi(g + 3)$, then M is isotopic to the standard embedding.*

Proof. If $\int_M |K| dA < 4\pi(g + 3)$, then it is elementary to show that there exists a straight line $L \subset \mathbb{R}^3$ such that if $P: \mathbb{R}^3 \rightarrow L$ is orthogonal projection on L , then $P|M$ is a Morse function with at most $2g + 4$ critical points (see Ref. [4] for a proof). However, the Morse inequalities imply that $P|M$ must have at most three critical points of index different from 1. Thus either $P|M$ or $(-P)|M$ is a Morse function with at most one critical point of index 2 and by the previous corollary M is standardly embedded in \mathbb{R}^3 .

COROLLARY 4. *Suppose M is an embedded compact surface contained in the unit ball $B^3 \subset \mathbb{R}^3$ such that $\gamma = (M \cap \partial B^3)$ is a Jordan curve which is the boundary of M . If there is no point $p \in M$ where the principal curvatures have the same sign and $|K_1(p)| \geq |K_2(p)| \geq 1$, then M is isotopic in B^3 , relative to γ , to any standard embedding.*

Proof. Consider the function $d: B^3 \rightarrow \mathbb{R}$ which is $d(x, y, z) = x^2 + y^2 + z^2$. After a small perturbation of M or d , the hypothesis in Corollary 4 implies that the function $d|M$ is a Morse function with no local maximums. The argument given in the proof of Corollary 2 implies that M is standardly embedded in B^3 up to isotopy in B^3 .

Remark. The Corollaries, 2–5 would also follow easily from the topological

uniqueness results of Morton in [14]. Morton's results were found independently and are based on the topological uniqueness of Heegard splittings of the three sphere S^3 . The first topological uniqueness result of the type proved here was given by Langevin and Rosenberg in [4].

For more general applications the following corollary is useful.

COROLLARY 5. *Let M_1 and M_2 be diffeomorphic surfaces which satisfy the hypotheses of theorem 1 except that ∂M_1 and ∂M_2 need not be the same or be on the same parallel planes P_0 and P_1 . Then M_1 and M_2 are isotopic in \mathbb{R}^3 .*

Proof. Using the general argument in the proof of the Theorem 1 it is sufficient to prove that M_1 is isotopic of some fixed "standard" example. By the proof of Assumption 4 in Theorem 1 we may assume that M_1 satisfies Assumptions 4 and 5. Furthermore we may assume that M_1 appears as in Fig. 2 near the plane P_0 .

Suppose that $\partial M_1 = \{\gamma_1, \gamma_2, \dots, \gamma_n, \alpha\}$ and $\mathcal{D} = \{D_1, D_2, \dots, D_n\}$ are the respective disks with boundary curves γ_i . Let $K(M_1)$ be the number of elements in the set $S = \{(i, j) | i \neq j \text{ and } D_i \cap D_j \neq \emptyset\}$. If the collection of disks \mathcal{D} is not pairwise disjoint, then there an index i such that $D_{i+1} \subset D_i$ and the interior of D_{i+1} is disjoint from M_1 . From Fig. 2 we may assume that M near P_0 looks like cylinders over ∂M together with a tube T connecting the curve γ_{i+1} to the cylinder over γ_i .

By an ambient isotopy (that is not the identity outside the region R between P_0 and P_1) of M_1 which moves the tube T and fixes $\partial M - \{\gamma_{i+1}\}$ we get a new surface M'_1 satisfying the conditions of the corollary and such that $\partial M'_1 = (\partial M \cup \tilde{\gamma}_{i+1}) - \{\gamma_{i+1}\}$ with $\tilde{\gamma}_{i+1}$ being a small circle close to γ_i and disjoint from D_{i+1} . It follows that $K(M'_1) = K(M_1) - 1$.

We may apply the above argument and the induction principle to get a new surface \tilde{M}_1 isotopic to M_1 , satisfying the hypothesis of the corollary and the boundary of \tilde{M}_1 consists of pairwise disjoint disks in P_0 . The proof the Assumption 4 in Theorem 1 applies to \tilde{M}_1 to show that there exists a level preserving isotopy taking any such surface with boundaries on P_0 and P_1 to a standard such example. This proves the corollary.

§2. THE TOPOLOGICAL UNIQUENESS OF MINIMAL SURFACES

In this section we shall give a generalization of Corollary 4 in the previous section when the surface M is minimal and is contained in a sphere of positive mean curvature in \mathbb{R}^3 . While the results hold in greater generality, for simplicity we essentially restrict ourselves to subsets of \mathbb{R}^3 . We refer the reader to [5, 6] for similar arguments.

PROPOSITION 1. *Suppose M^n is a flat compact n -dimensional manifold with a piecewise smooth boundary. If the interior angles of all non-smooth part of the boundary are less than π and if the connected smooth parts of the boundary of M^n have either zero or strictly positive mean curvature, then the induced map*

$$i_*: \pi_1(\partial M^n) \rightarrow \pi_1(M^n)$$

is onto unless the boundary is totally geodesic.

Proof. In the case that M^n has strictly positive Ricci curvature, a proof of the

above theorem was given by Lawson[5] in the stronger sense that $i_*: \pi_1(\partial M^n) \rightarrow \pi_1(M^n)$ is always onto. His use of a variation argument due to Synge does not work in the case of nonnegative Ricci curvature. In fact, $i_*: \pi_1(\partial M^n) \rightarrow \pi_1(M^n)$ need not be onto when the boundary is totally geodesic.

The condition that $i_*: \pi_1(\partial M^n) \rightarrow \pi_1(M^n)$ be onto is equivalent by elementary covering space theory to showing that the boundary $\partial \tilde{M}^n$ of the universal covering space \tilde{M}^n of M^n is path connected. By the compactness of M^n and the application of the covering transformations which are isometries in the pulled back Riemannian metric, one can show that if $\partial \tilde{M}^n$ is not path connected, then (1) there are two path components ∂_1 and ∂_2 of the boundary $\partial \tilde{M}^n$ such that ∂_2 is a closest boundary component to ∂_1 , where the distance between two path components is defined to be the infimum of lengths of all piecewise smooth paths joining the boundary components; (2) there is a geodesic $\gamma: [0, d] \rightarrow M^n$ which realizes the minimal distance between ∂_1 and ∂_2 .

Suppose $\gamma(0) = p \in \partial_1$ and $\gamma(d) = q \in \partial_2$. Since M^n is flat and γ is locally a straight line in a ball in \mathbb{R}^n , a unique continuation argument proves that an ϵ -neighborhood N of γ embeds isometrically in \mathbb{R}^n . Since the interior angles at the non-smooth parts of the boundary of \tilde{M}^n have interior angles less than π , we can be sure that p and q are regular points on the boundary. Hence for a sufficiently small ϵ we have $N \cap \partial_1$ and $N \cap \partial_2$ are smooth connected parts of the boundary.

Now consider N as a subset of \mathbb{R}^n with $\gamma(t) = (0, 0, \dots, 0, t)$. Now translate $V = N \cap \partial_1$ along the straight line $\gamma(t)$ until it intersects $W = N \cap \partial_2$. Since $p = \gamma(0)$ and $q = \gamma(d)$ are points of least distance between V and W , the translated surface $V' = V + (0, 0, \dots, d)$ intersects W at the point $q = (0, 0, \dots, d)$ and near q we have that W is a graph over V' . Note that the mean curvature of V' with respect to the inward normal to N at q is the negative of the corresponding point p on V . This is because translation is an isometry and the new normal vector field is the negative of the previous.

If the mean curvature $K_{V'}(q)$ is negative or if the mean curvature $K_W(q)$ is positive, then elementary arguments (see Lemma 1 in §3 of Ref. [7]) give rise to a contradiction. Hence we may assume that $K_{V'} \equiv 0 \equiv K_V$ and $K_W \equiv 0$.

The maximum principle for minimal hypersurfaces implies that V' and W' are equal in a small neighborhood of q' . This minimum distance hypothesis implies that the lines parallel and close to γ are all minimizing geodesics and hence are perpendicular to V and W at all points. This implies that V and W are parts of parallel hyperplanes.

The discussion in the previous paragraph implies that a neighborhood of p_1 on ∂_1 and a neighborhood of p_2 on ∂_2 are totally geodesic and "parallel". In fact a continuation argument implies the entire piecewise smooth region R of ∂_1 which contains the point p is totally geodesic and of distance d from ∂_2 . Furthermore the minimal distance from points on R to ∂_2 are achieved by a geodesic that is a parallel translate of the geodesic γ .

Therefore a nonsmooth corner of the region R also has distance d from ∂_2 . Since the interior angles on the corner points of ∂_1 are less than π , the minimum distance between ∂_1 and ∂_2 can not occur at such a corner point. Thus R must be all of ∂_1 . It also follows that \tilde{M} is naturally isometric with $\partial_1 \times [0, d]$ with the product metric. This shows that the boundary of \tilde{M} is totally geodesic contradicting our hypothesis and hence proves the theorem.

Remark 1. The Möbius strip M with the flat metric is a good example of a flat manifold with totally geodesic boundary such that the image of the fundamental group of the boundary in the fundamental group of the Möbius strip has index two. A

more careful examination of the proof of the previous theorem shows that if the fundamental group of a totally geodesic boundary does not map onto the fundamental group of M^n , then the image of the fundamental group has index two. For an orientable example one may consider a twisted interval bundle over the Klein bottle.

Definition. Suppose $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ are a collection of disjoint Jordan curves on the boundary of a three dimensional manifold M^3 . Let H be a compact connected embedded surface in M^3 with boundary curves Γ . If H disconnects M^3 into two pretzels or doughnuts, then H will be called a Heegard splitting or surface for M^3 with boundary curves in Γ .

PROPOSITION 2. *Suppose M is an embedded connected compact minimal surface with boundary contained in a ball in \mathbb{R}^3 with positive mean curvature on the boundary. If the boundary of M is a collection $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ of smooth Jordan curves on the boundary sphere, then M is a Heegard splitting of the ball with boundary curves in Γ .*

Proof. In [7, 9] it is shown that M is transverse to the boundary of the ball. Elementary topological arguments imply that M disconnects the ball into two compact three dimensional manifolds M_1 and M_2 with boundaries satisfying the hypotheses in the previous theorem. Therefore the fundamental groups of the boundaries map onto the fundamental groups of the corresponding pieces. By Alexander's theorem M_1 and M_2 do not contain a fake three ball. A theorem of Papakyriakopoulos [18] now shows that M_1 and M_2 are pretzels. This proves the proposition.

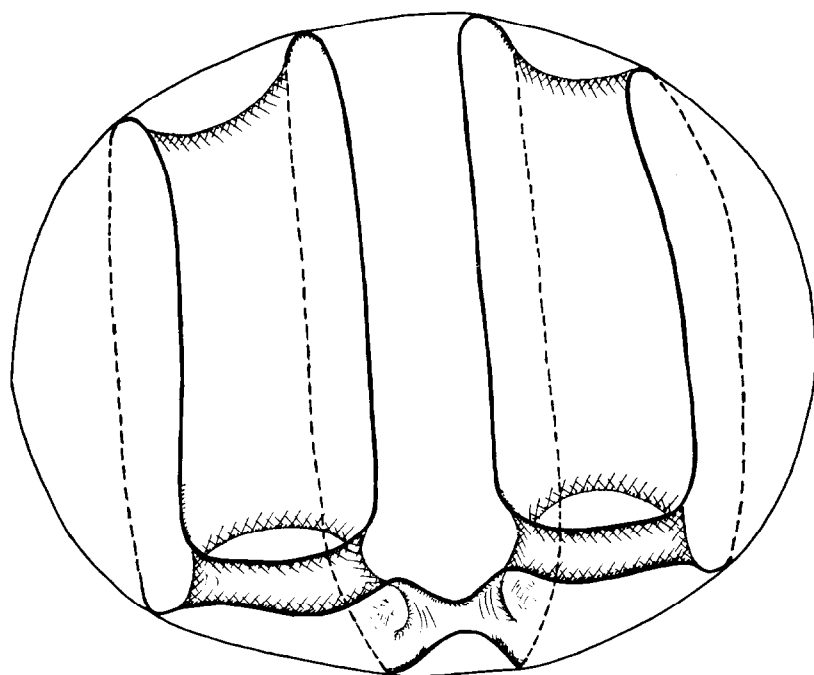
In the case M is an embedded minimal surface in S^3 , $S^2 \times S^1$ with the product metric, or a flat torus T^3 , where M is not totally geodesic, it can be proved that M disconnects the three manifolds into pretzels ([5, 6]). In the case of S^3 or $S^2 \times S^1$ Waldhausen has proved the deep fact that these decompositions or Heegard splittings are topologically unique up to isotopy. Hence embedded minimal surfaces of a given genus are isotopic in S^3 and $S^2 \times S^1$.

Combining Waldhausen's technique together with the above proposition, we can prove the following very beautiful uniqueness theorem.

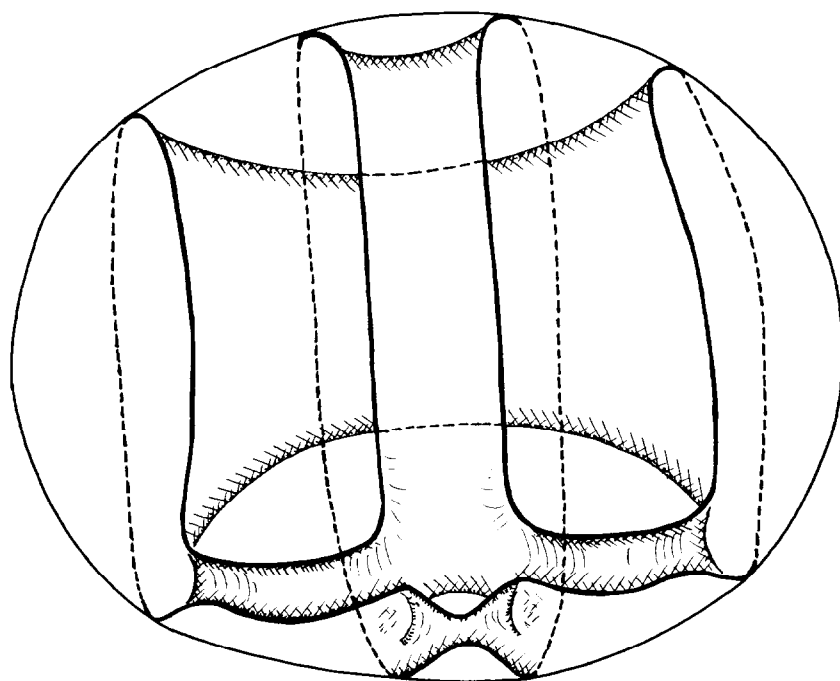
THEOREM 2. *Suppose γ is a smooth Jordan curve in \mathbb{R}^3 which lies on the boundary of a smooth three ball B^3 with positive mean curvature. If $f: M \rightarrow B^3$ is an embedded minimal surface, then M is standardly embedded in B^3 up to isotopy. In particular if M_1 and M_2 are two embedded homeomorphic minimal surfaces in B^3 with boundary γ , then there is a homeomorphism $h: B^3 \rightarrow B^3$ with $h(M_1) = M_2$ and $h|_{\partial B^3} = id_{\partial B^3}$.*

Proof. By the previous proposition M_1 gives rise to a Heegard splitting of the ball B^3 which has one boundary curve. It follows directly from the discussion ([20], p. 202) that such Heegard splittings are unique up to isotopy, which proves the theorem.

Example 1. In Figs. 5(a) and (b) are two examples of stable minimal surfaces which bound the same Jordan curve on the unit sphere S^2 and which have the same topology. These examples arise from joining parts of "catenoids" by thin bridges (see [9]). For fun the reader can try to prove directly that the two minimal surfaces are isotopic.



(a)



(b)

Fig. 5.

Example 2. The following “knotted” example of a minimal surface with boundary being three Jordan curves on the boundary of a convex box demonstrates that even though the minimal surface gives a Heegard splitting to the box, the surface need not be standardly embedded when there is more than one boundary curve. This example is constructed from three stable catenoids joined by thin bridges.

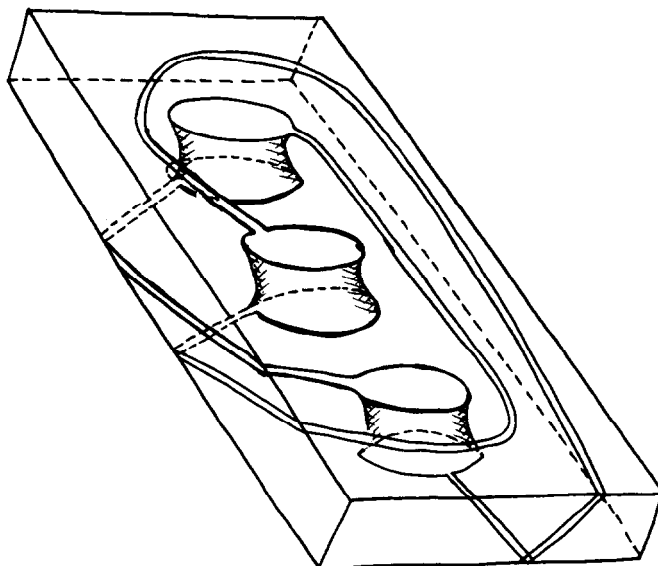


Fig. 6.

§3. THE TOPOLOGY OF THE COMPLEMENTS OF CERTAIN PROPER EMBEDDED CODIMENSION-ONE SUBMANIFOLDS OF \mathbb{R}^n

THEOREM 3. *Let M be a complete proper codimension-one submanifold of \mathbb{R}^n which has non-positive sectional curvature with respect to the induced metric. If N is a component of $\mathbb{R}^n - M$, then N is diffeomorphic to an open regular neighborhood of a proper one-dimensional CW-subcomplex in \mathbb{R}^n .*

After a translation of M , we may suppose that the origin of \mathbb{R}^n is contained in the interior of N . Consider a small smooth perturbation $d: \mathbb{R}^n \rightarrow \mathbb{R}$ of the distance function to the origin such that the level sets $d^{-1}(t)$ consist of strictly convex spheres for $t > 0$ and $d^{-1}(0)$ is the origin. Furthermore, we shall consider a perturbation d which is a Morse function on M . We now examine the indices of the critical points of $d|_M$.

LEMMA 1. *The indices of the critical points of $d|_M$ are zero and one.*

Proof of the lemma. The proof of the lemma will be by contradiction. Suppose there exists a point p that is a critical point of $d|_M$ of index greater than 1. It follows from the definition of the index that there is a two dimensional subspace $V \subset T_p M$ such that the Hessian of $d|_M$ at p is negative definite.

Let W be the three dimensional affine subspace in \mathbb{R}^n generated by V and the normal space at p . Since the Hessian on V is negative, the surface $X = W \cap M$ near p has strictly positive Gaussian curvature considered as a surface in the three space W . Thus, the sectional curvature of the plane $V = T_p X$ is positive on the two dimensional submanifold X of M .

On the other hand, the second fundamental form of X in M is identically zero at the point p . The Gauss formula implies in this case that the sectional curvature of V in $T_p M$ equals the sectional curvature of V as a subspace of the tangent space to X at p which is positive. This contradiction proves the lemma.

Proof of the theorem. Consider the function d on the closure \bar{N} of N in \mathbb{R}^n . It is clear that the topology of $\bar{N}_r = (d|_{\bar{N}})^{-1}((-\infty, r])$ changes precisely at the critical points of $d|_M$. The analysis of how the topology of N_r changes at a nondegenerate critical point of d is well known and can be described as follows.

Let T be the outward normal on N at a critical point p of index k of $d|_N$. If $\langle T, \text{grad}_p(d) \rangle$ is negative, then the topology of \bar{N} , upon passing the critical point p , changes by attaching a k handle. If $\langle T, \text{grad}_p(d) \rangle$ is positive, then the topology of \bar{N} does not change upon passing the point p .

Lemma 1 together with the above description of how the topology changes shows that N is diffeomorphic to a regular neighborhood of a one-dimensional proper CW-subcomplex in \mathbb{R}^n . This completes the proof of the theorem.

Remark. It is straightforward to see that the proof of Theorem 3 holds in the following greater generality. Suppose $\langle \cdot, \cdot \rangle$ is a complete metric on \mathbb{R}^n and suppose that for every p in \mathbb{R}^n there exists a unique geodesic joining p to the origin. Suppose also that the balls B_r of distance r from the origin are strictly convex. Let Δ be the infimum of the sectional curvatures of ∂B_r , and suppose Δ is finite. Let M be a complete proper embedded codimension-one submanifold of \mathbb{R}^n . If the sectional curvature of M is less than or equal to Δ , then the result in Theorem 3 holds when the origin of \mathbb{R}^n is an element of M .

For applications it's important to note that the assumption of convexity of ∂B_r implies Δ is greater than or equal to the infimum of the sectional curvatures of \mathbb{R}^n with the metric $\langle \cdot, \cdot \rangle$. Thus, for example, if $\langle \cdot, \cdot \rangle$ has sectional curvature C where $K \leq C \leq 0$, then the balls B_r are convex and we may apply the proof of the theorem for M having sectional curvatures $C_M \leq K$. If C is constant, then the sectional curvatures of ∂B_r are greater than zero so the result holds for $C_M \leq 0$.

The following corollary gives an important case of the above theorem and the generalization discussed in the remark.

COROLLARY. *Let $\langle \cdot, \cdot \rangle$ be a complete metric on \mathbb{R}^3 with non-positive sectional curvature. If M is a complete proper embedded minimal surface in \mathbb{R}^3 , then M disconnects \mathbb{R}^3 into regions that are diffeomorphic to regular neighborhoods of proper one dimensional CW-subcomplexes in \mathbb{R}^3 .*

Proof of the corollary. From the above discussion it is sufficient to know that the sectional curvature at a point p of M is less than or equal to the sectional curvature of the tangent plane $T_p M$ in \mathbb{R}^3 . This calculation follows directly from minimality and the Gauss formula which calculates the difference of sectional curvatures in terms of the second fundamental form on M .

Remark. The above corollary can be generalized somewhat to include another interesting geometric case. Let $\langle \cdot, \cdot \rangle$ be a complete metric on \mathbb{R}^3 with non-positive sectional curvature. If M has non-negative mean curvature as the boundary of a region R of \mathbb{R}^3 , then R is a regular neighborhood of a one-dimensional CW-subcomplex contained in \mathbb{R}^3 . The proof of this generalization of the corollary is similar

to the proof of Theorem 3. It is sufficient to note that the perturbed distance function d in the lemma for Theorem 3 has no local maximum on the complement of R . This fact follows from an elementary geometric comparison of mean curvatures (see, e.g. ([7], p. 19, Lemma 1).

§4. THE TOPOLOGICAL PLACEMENT OF PROPER EMBEDDED MINIMAL SURFACES IN \mathbb{R}^3 WHICH ARE OF FINITE TOPOLOGICAL TYPE

THEOREM 4. *Let M be a complete proper embedded surface in \mathbb{R}^3 which is diffeomorphic to a compact surface punctured in a finite number of points. Suppose that the distance function $d: \mathbb{R}^3 \rightarrow \mathbb{R}$ from the origin restricted to M is a Morse function with critical points of indice 0 and 1. Then there exists a large round ball B centered at the origin such that:*

- (1) *There exists a component X of $B \cap M$ such that $M - X$ consists of annular ends of M .*
- (2) *M is isotopic relative to X to a new surface M' which satisfies*
 - (i) *$M' \cap B = X$.*
 - (ii) *The ends of M' are standardly embedded up to isotopy in $\mathbb{R}^3 - \mathring{B}$ relative to the curves $M' \cap \partial B$.*

Proof. For any compact subset \tilde{M} of M there exists a regular value r of $d|_M$ so that \tilde{M} is contained in the ball B_r of radius r centered at the origin. Letting \tilde{M} be a compact subset of M whose complement consists of annular ends of M , there exists a large ball B in \mathbb{R}^3 centered at the origin such that $\tilde{M} \subset \mathring{B}$ and M is transverse to ∂B . Let X be the connected component of $B \cap M$ which contains \tilde{M} .

ASSERTION 1. *$M - \mathring{X}$ consists of closed annular ends of M .*

Proof of Assertion 1. This assertion is equivalent to proving that $X - \mathring{M}$ consists of closed annuli having one boundary curve on X and one boundary curve on \tilde{M} . Since X is the connected component of $B \cap M$ containing \tilde{M} and every component C of $X - \mathring{M}$ is contained on an annular end of M , the component C has one boundary curve which is the boundary curve of an annular end and at least one boundary curve on ∂X .

If the component C has more than two boundary curves, then there is a boundary curve in C which bounds a disk D on M . As this disk leaves the ball B , the distance function d has a local maximum on D which contradicts the assumptions on d in the statement of the theorem. This contradiction implies the assertion.

A similar argument proves the following assertion.

ASSERTION 2. *Let C be a component of $M - \mathring{X}$. If $r = \text{radius of } B$ and $R > r$ is a regular value of $d|_M$, then $W = (d|_C)^{-1}([0, R])$ has a unique component with boundary curve ∂C and this component is an annulus. The other components of W are disks.*

With these preliminaries we are in a position to construct the required isotopy. For convenience we shall assume that the critical points of $d|_M$ have distinct critical values. Let $W = M - \mathring{X}$ and $c_1 < c_2 < \dots < c_n < \dots$ be the critical values of $d|_W$. After a diffeomorphism of the range of d , we may assume that the critical points of $d|_W$ lie sequentially at the values $10, 20, \dots, 10n, \dots$

Consider the Jordan curve

$$\Gamma_n = \{\alpha_1^n, \alpha_2^n, \dots, \alpha_{k_n}^n\} \subset (d|W)^{-1}(10n+1)$$

which are the boundaries of the disks in Assertion 2 for $R = 10n + 1$. Let $\theta_n = \{\theta_1^n, \dots, \theta_j^n\}$ be the annular regions in assertion 2 for $R = 10n + 1$. By the Jordan curve theorem the curves in Γ_n bound disks in the sphere $S_n = \partial B_n$ where $B_n = d^{-1}((-\infty, 10n + 1])$.

ASSERTION 3. *Every $\gamma \in \Gamma_n$ is the boundary of a unique disk $D(\gamma)$ in $S_n - \theta_n$.*

Proof of Assertion 3. Since we may assume that θ_n is non-empty, uniqueness is clear. On the other hand, γ is the boundary of a disk $\tilde{D}(\gamma)$ in B_n which is disjoint from the component M_n of $M \cap B_n$ containing X . By Alexander's theorem $\tilde{D}(\gamma)$ disconnects B_n into two balls. Let $B_n(\gamma)$ be the ball which does not contain M_n . The intersection of $B_n(\gamma)$ with S_n is the required disk $D(\gamma)$.

ASSERTION 4. *Let M_n be as above. Then there is an isotopy M' of M that fixes M_{n-1} and is the identity outside of B_n . Furthermore, $d|M'$ has no critical points of index 2 and $M' \cap B_{n-1} = M_{n-1}$.*

Proof. We will use the notation in the proof of Assertion 3. For every curve γ in Γ_n replace the disk $\tilde{D}(\gamma)$ by $D(\gamma)$ pushed into a small neighborhood of ∂B_n in B_n in such a way that the new surface M' obtained is smooth and d restricted to the pushed in disk has a unique critical point and this critical point has index 0. The proof that M' is isotopic to M follows by induction on the number of curves in Γ_n and uses the fact that $\tilde{D}(X)$ and $D(\gamma)$ bound balls. Starting with a smallest such ball, the replacing of disks involved reduces the number of curves to be considered (see also Assertion 2 in the proof of Theorem 1). Since these constructions are more or less standard, we leave the details to the reader to check. This proves the assertion.

We now can apply Assertion 4 inductively starting with $n = 2$ to get a surface M' , then apply the assertion to M' for $n = 3$ to get a new surface M'' . Since every point p in M is contained in M_n for n sufficiently large, the n th isotopy I_n of M leaves the point $I_{n-1}(p)$ fixed. Thus the isotopies inductively applied to M in the construction of the primed surfaces $M'' \cdots'$ converge on M to yield a new surface \tilde{M} .

Let $\Delta_n = B_n - \hat{B}_{n-1}$. The argument of case 2 in the proof of Assertion 2 of Theorem 1 shows that after an isotopy of $\tilde{M} \cap \Delta_n$ the new \tilde{M} has no critical points in Δ_n . Composing all of the above isotopies to M yields an M' such that M' satisfies the items (1) and (2) (i) of the theorem. Since $d|M'$ has no critical points outside B , $(\mathbb{R}^3 - \hat{B}) - M'$ has a product structure. As product structures are unique, the ends of M' are standardly embedded in $\mathbb{R}^3 - \hat{B}$ relative to $M' \cap \partial B$ (see the proof of Theorem 4 in Ref. [21] for this type of application of product structures). This completes the proof of the theorem.

COROLLARY. *Suppose $\langle \cdot, \cdot \rangle$ is a complete metric on \mathbb{R}^3 with non-positive sectional curvature. Let M be a complete proper embedded minimal surface in \mathbb{R}^3 which is diffeomorphic to a compact surface punctured in a finite number of points. Then*

- (1) *If M has one end, then M is standardly embedded in \mathbb{R}^3 . In particular, two such simply connected examples are isotopic.*
- (2) *If M is diffeomorphic to an annulus, then M is isotopic to the catenoid.*

Proof of the corollary. Let B be the ball in the theorem. By the corollary to Theorem 1, the surface $X = M \cap B$ is standardly embedded in B if M has one end. If M is an annulus, then X gives rise to a Heegard splitting of B . Such Heegard splittings of the ball are standard up to isotopy by the usual application of Dehn's lemma. The above theorem shows that the ends of M are standardly embedded in $\mathbb{R}^3 - B$. Thus M is standardly embedded up to isotopy which completes the proof of the corollary.

Remark 1. As discussed in the remark after Theorem 3 the above corollary holds in much greater generality. For example one can replace "minimal" by "nonpositive Gaussian curvature" in the case the metric $\langle \cdot \rangle$ is flat. It follows from this remark and also directly from Theorem 3 that the following proper surface is not isotopic to a surface with nonpositive curvature. This example corresponds to the plane connected sum with a knotted torus. (See figure 7).

Remark 2. Until now the only known examples of complete embedded proper minimal surfaces in \mathbb{R}^3 which are diffeomorphic to compact surfaces punctured in a finite number of points are the plane, the catenoid and the helicoid. Thus one of the outstanding problems in the classical theory of minimal surfaces is to find new examples.

Complete minimally immersed surfaces with finite total curvature are all conformally equivalent to compact Riemann surfaces punctured in a finite number of points by a theorem of Osserman[17]. A rather complete topological study of these surfaces was carried out in [3].

In conclusion the author would like to pose three problems concerning the topological uniqueness of minimal surfaces in \mathbb{R}^3 . He refers the reader to [7] for a discussion of these and other problems. A positive solution to Problem 1 gives a positive solution to Problem 2.

Problem 1. Suppose Γ is a collection of disjoint Jordan curves on $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. If M_1 and M_2 are embedded homeomorphic compact

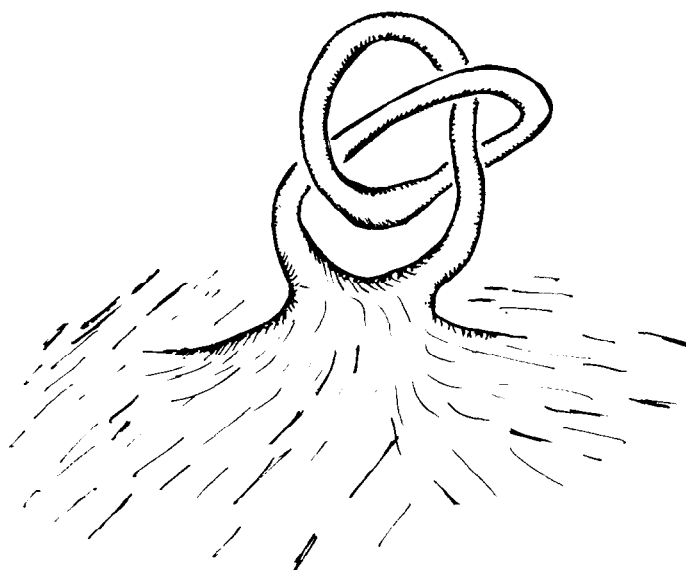


Fig. 7.

connected minimal surfaces in the unit ball B which have boundary Γ , then M_1 and M_2 are isotopic in B .

Problem 2. If M_1 and M_2 are embedded triply periodic minimal surfaces in \mathbb{R}^3 or M_1 and M_2 are embedded complete proper minimal surfaces of \mathbb{R}^3 which are diffeomorphic to a compact surface punctured in a finite number of points, then M_1 and M_2 are isotopic in \mathbb{R}^3 .

Problem 3. Define an immersed surface to be locally tight if it is a local minimum for total curvature in the space of immersions. If M is an embedded locally tight closed surface in \mathbb{R}^3 , then M is tight. In particular by Corollary 3 to Theorem 1, M is standardly embedded.

Acknowledgement—The author would like to express his gratitude to John Douglas Moore who proofread the first version of this material.

REFERENCES

1. F. J. ALMGREN, JR and L. SIMON: Existence of embedded solutions to Plateau's problem. *Ann. Scuola Norm. Sup. Pisa* VI (1979), 447–495.
2. W. HAKEN, Some results on surfaces in 3-manifolds. *MAA Studies in Mathematics*, Vol. 5.
3. L. P. JORGE and W. H. MEEKS III: The topology of complete minimal surfaces of finite total Gaussian curvature. preprint.
4. R. LANGEVIN and H. ROSENBERG: On curvature integrals and knots. *Topology* 15 (1976), 405–416.
5. H. B. LAWSON: The unknottedness of minimal embeddings. *Invent. Math.* 11 (1970), 183–187.
6. W. H. MEEKS III: The geometry and conformal structure of triply periodic minimal surfaces in \mathbb{R}^3 . *Doctoral Thesis* (revised). University of California, Berkeley (1975).
7. W. H. MEEKS III: Lectures on Plateau's Problem. *IMPA*. Rio de Janeiro, Brazil (1978).
8. W. H. MEEKS III and S. T. YAU: The classical Plateau problem and the topology of three dimensional manifolds. to appear.
9. W. H. MEEKS III and S. T. YAU: The existence of embedded minimal surfaces and the problem of uniqueness. preprint.
10. W. H. MEEKS III and S. T. YAU: The equivariant Dehn's lemma and loop theorem. preprint.
11. W. H. MEEKS III and S. T. YAU: Decomposition and classification of group actions on open three-dimensional manifolds. in preparation.
12. W. H. MEEKS III and S. T. YAU: Compact group actions on \mathbb{R}^3 . *Conf. on the Smith Conjecture at Columbia University*, Columbia University, New York (1979).
13. W. H. MEEKS III and S. T. YAU: Topology of three dimensional manifolds and embedding problems in minimal surface theory. *Ann. Math* (1980).
14. H. MORTON: A criterion for an embedded surface in \mathbb{R}^3 to be unknotted. to appear in *Proc. of Topology Conference in Sussex*, England (1977).
15. J. MILNOR: Morse Theory. *Ann. Math. Studies*, Vol. 51. Princeton University Press (1970).
16. J. C. C. NITSCHKE: Contours bounding at least three solutions to Plateau's problem. *Arch. Rational Mech. Anal.* 30, (1968), 1–11.
17. R. OSSERMAN: *A Survey of Minimal Surfaces*. Van Nostrand Reinhold Company, New York, New York (1969).
18. C. D. PAPAKYRIAKOPOULOS: On solid tori. *Proc. London Math. Soc.* VII (1957), 281–299.
19. J. SINGER: Three dimensional manifolds and their Heegard diagrams. *Trans. Am. Math. Soc.* 35 (1933), 88–111.
20. F. WALDHAUSEN: Heegard splittings of the 3-sphere. *Topology* 7 (1968), 195–203.
21. The Proof of the Smith Conjecture. *Proc. of Conference at Columbia University*. Columbia University (1979) in preparation.

Stanford University
Stanford, CA, 94305, U.S.A.